[19] Bennett, J. A., "Ultraharmonic Motion of a Viscously Damped Nonlinear Beam," *AIAA Journal*, Vol. 11, No. 5, May 1973, pp. 710–715.

[20] Tseng, W.-Y. and Dugundji, J., "Nonlinear Vibrations of a Buckled Beam under Harmonic Excitation," *Journal of Applied Mechanics*, Vol. 38, 1971, pp. 467–476.

[21] Nayfeh, A. H., Mook, D. T., and Sridhar, S., "Nonlinear Analysis of the Forced Response of Structural Elements," *Journal of the Acoustical Society of America*, Vol. 55, No. 2, 1974, pp. 281–291.

[22] Ray, J. D. and Bert, C. W., "Nonlinear Vibrations of a Beam with Pinned Ends," *Transactions of the ASME, Journal of Engineering for Industry*, Vol. 91, Ser. B, 1969, pp. 997–1004.

[23] Nayfeh, A. H., Mook, D. T., and Marshall, L. R., "Nonlinear Coupling of Pitch and Roll Modes in Ship Motions," *Journal of Hydronautics*, Vol. 7, No. 4, Oct. 1973, pp. 145–152.

[24] Mook, D. T., Marshall, L. R., and Nayfeh, A. H., "Subharmonic and Superharmonic Resonances in the Pitch and Roll Modes of Ship Motions," *Journal of Hydronautics*, Vol. 8, No. 1, Jan. 1974, pp. 32–40.

[25] Timoshenko, S. P., *Vibration Problems in Engineering*, 3rd ed., Van Nostrand, Princeton, N.J., 1956.

[26] Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.

[27] Martin, R. S. and Wilkinson, J. H., "The Implicit QL Algorithm," *Numerische Mathematik*, Vol. 12, 1968, pp. 377–383.

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Prediction of Unsteady Airloads for Oblique Blade-Gust Interaction in Compressible Flow

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The techniques of Galilean-Lorentz transformation and matched asymptotic expansions are used to simplify the procedure of calculating the lift and pressure distribution induced on an infinite-span thin wing interacting with an oblique sinusoidal gust in subsonic flow. This technique requires that the product of the flow Mach number and the reduced frequency is small. Under this condition, the inner region of the transformed space behaves as an incompressible flow so that existing incompressible flow theories can be used as a basis to construct closed-form solutions for the airload induced on the wing. This approach is an extension of the GASP approximation developed by Sears and Amiet. Results are obtained for both the magnitude and the phase of the unsteady lift due to interaction with gust. These results are compared with available numerical results. Some discrepancies are noted and discussed.

Nomenclature

a	= speed of sound
b_o	= wing semichord
b	= wing semichord in the Galilean-Lorentz (G-L) transformed space
$C(S)$	= Theodorsen function $[=H_1^{(2)}(S)/\{H_1^{(2)}(S)+iH_0^{(2)}(S)\}]$
C_L	= lift coefficient
C_p	= pressure coefficient
g	$=\{1-(\tan\Lambda^T/M)^2\}^{1/2}$
g'	$=-ig$
$H_n^{(1,2)}(x)$	= nth order Hankel function of the first or the second kind
$I_n(x)$, $K_n(x)$	= nth order modified Bessel functions of the first and the second kind
$J_n(x)$	= nth order Bessel function of the first kind
k	= gust wave number
k_o	= normalized gust wave number $(=kb_o)$
k_x, k_y	= chordwise and spanwise components of the gust wave number
k_2	$=k_o\sin\Lambda$
K, K_o, K_X, K_Y, K_2	= counterparts of k, k_o, k_x, k_y, k_2 in the G-L transformed space
K'	$=K_2+i\sigma$
M	= freestream Mach number
M_c	= airload convection Mach number $(=M/\sin\Lambda)$
P	= amplitude of the pressure coefficient in the G-L transformed space
r_o, θ	= cylindrical coordinates in the outer region
$\bar{r}$	= cylindrical coordinate in the inner region $(\bar{r}=r_o/\varepsilon)$
s	= reduced frequency $(=\omega b_o/U)$
S	= reduced frequency in the G-L transformed space $(=\Omega b/U=s/\beta^2)$
T_A	= aerodynamic transfer function $(=C_L/\{2\pi\alpha_o/\beta\})$
T_o	= aerodynamic transfer function for the incompressible flow
$T_{o_{2D}}$	= aerodynamic transfer function for the two-dimensional incompressible flow
U	= freestream velocity
w	= gust upwash velocity
x, y, z, t	= coordinates of the original space attached to the wing
x', y', z', t'	= coordinates of the Galilean transformed space convected with the freestream
X, Y, Z, T	= coordinates of the G-L transformed space attached to the wing
$\bar{X}, \bar{Y}, \bar{Z}, \bar{T}$	= normalized inner variables in the G-L transformed space
X_o, Y_o, Z_o, T_o	= normalized outer variables in the G-L transformed space
α	= incident angle
α_o	= incident angle at the midchord

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β = $(1-M^2)^{1/2}$
ε = small parameter for expansion ($=SM$)
η = $\bar{X}+i\bar{Z}$
Λ = wing-gust interaction angle
λ_a = acoustic wavelength
σ = $(M/\beta)^2 s$
ϕ = perturbation velocity potential
Φ = normalized perturbation velocity potential
ϕ_∞ = phase of the aerodynamic transfer function
ω = radian frequency
Ω = radian frequency in the G-L transformed space
$\hat{(\ )}$ = amplitude
$(\ )^i$ = inner solution
$(\ )^{io}$ = outer expansion of the inner solution
$(\ )^o$ = outer solution
$(\ )^{oi}$ = inner expansion of the outer solution
$(\ )^T$ = property for the G-L transformed space
$(\ )^*$ = conjugate of a complex function
$(\ )_n$ = nth order solution
$(\ )_o$ = property for the incompressible flow

Introduction

THE evaluation of the unsteady load distribution induced on a thin wing due to interaction with turbulent gusts has always been an important problem in aerodynamics because of its broad application to the calculation of dynamic loads on fixed and rotary wings, unsteady loads due to rotor-stator interaction and the prediction of aerodynamic noise from unsteady forces. Since for small disturbances the governing equation is linear, it is possible through Fourier analysis to separate the response to any three-dimensional gust into the response to a spectrum of oblique sinusoidal gusts.

One of the classical problems in this category is the two-dimensional wing-gust interaction problem solved analytically by Sears[1] for incompressible flow. He considered a two-dimensional wing interacting with a sinusoidal gust convected with the freestream having its wave front parallel to the leading edge of the wing. Graham[2] treated the corresponding three-dimensional problem for incompressible flow by using a numerical-series method. His model has the same two-dimensional wing, but the wave front of the gust makes an oblique angle to the leading edge. Filotas[3] solved the same three-dimensional incompressible-flow problem analytically and derived approximate closed-form expressions for the lift and pressure distribution. For wing-gust interactions in compressible flow, Graham[4] developed a similarity rule for the three-dimensional problem, but gave only an exact numerical solution for the two-dimensional case to be used in conjunction with his three-dimensional incompressible-flow solution. Johnson[5] approached the problem numerically starting directly from the general compressible-flow equation. He showed that the general problem of wing-gust interactions had an elliptic, transitional, or hyperbolic character corresponding to the convective speed of the airload along the gust wave front being subsonic, transonic, or supersonic, respectively.

By using a Galilean-Lorentz transformation and matched asymptotic expansions, Amiet and Sears[6] developed a technique called the "GASP" approximation to simplify the compressible-flow problem to an equivalent incompressible-flow problem in the inner region, under the condition of large acoustic wavelength-to-chord ratio. With this technique, Osborne[7] modified Sears' two-dimensional, incompressible solution for the compressible flow in a simple analytic form. However, it has once been pointed out by Miles[8] and has recently been remarked again by Amiet[9] that the GASP approximation cannot be applied to the two-dimensional airfoil problem as Osborne did, "since the solution for ϕ would involve integration over the infinite span, and this integration generally would not be commutative with the assumed expansion in powers of the frequency."[8]

In the present analysis, the GASP approximation is used to derive a similarity rule for the three-dimensional interaction of a two-dimensional wing and convected oblique sinusoidal gusts in subsonic flow. The wing span here is also infinite, but the finite spanwise gust wavelength can be considered in a sense comparable to a finite span. Hence the anomalous two-dimensional effect indicated by Miles[8] and Amiet[9] should not be a problem. Even though the usefulness of the analytic results is somewhat limited by the requirement that the product of the Mach number and the reduced frequency should be small, they are expected to serve as bases for the check of the more accurate numerical calculations. In this approach, a Galilean-Lorentz transformation is first used to reduce the governing equation to the ordinary wave equation with an accompanying modification of the boundary conditions. Then analytic solutions of series form are obtained through an application of the method of matched asymptotic expansions with the small parameter being the product of the flow Mach number and the reduced frequency. The lowest approximation for the inner region is found to be either a three-dimensional or a two-dimensional incompressible-flowfield depending upon the flow Mach number and the wing-gust interaction angle. Therefore, Sears' two-dimensional and/or Filotas' three-dimensional incompressible-flow solutions can be used as bases to construct analytic expressions for the airload distribution of all orders.

The analytic results from the present analysis are compared with Johnson's numerical solutions for the general three-dimensional wing-gust interactions in subsonic flow. The agreement is very good for the elliptic region. However, some discrepancies in the hyperbolic region are noted and discussed.

Formulation of the Theoretical Model

The theoretical model chosen here consists of a rigid, flat, thin wing of infinite span and chord width $2b_o$, flying with speed U through an oblique sinusoidal gust that is stationary in the fluid. The gust is characterized by the wave number k and the interaction angle Λ between its wave front and the wing leading edge (Fig. 1). The upwash velocity of the sinusoidal gust can be written as

$$w(x, y, t) = \hat{w}\, e^{i(k_x x + k_y y - \omega t)} \tag{1}$$

with $k_x = k\cos\Lambda$, $k_y = k\sin\Lambda$ being the wave number components in x direction, y direction and $\omega = kU\cos\Lambda$ being the radian frequency.

For small perturbations, the velocity field due to the thin wing is irrotational and may be linearly combined with the gust velocity field, even though the gust flow is vortical. In terms of the perturbation velocity potential function $\phi(x, y, z, t)$, the total velocity components in the flowfield are $U+\phi_x$, ϕ_y and $w+\phi_z$. The linearized governing equation for the flow is

$$\nabla^2\phi - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \phi = 0 \tag{2}$$

and the perturbation pressure coefficient is given by

$$C_p = -\frac{2}{U}\left(\frac{\partial\phi}{\partial x} + \frac{1}{U}\frac{\partial\phi}{\partial t}\right) \tag{3}$$

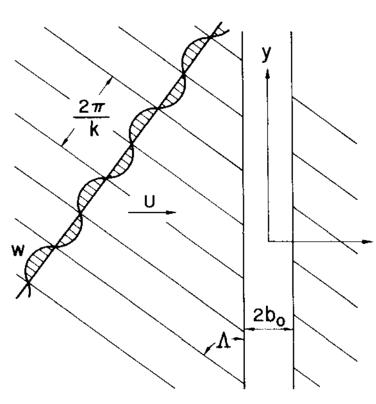

Fig. 1 Infinite-span thin wing interacting with a sinusoidal gust.

The boundary conditions are as follows:

1) The flow is tangent to the wing surface,

$$\phi_z|_{z=0\pm} = -\hat{w}e^{i(k_x x + k_y y - \omega t)} \quad \text{for} \quad |x| \leqq b_o \tag{4}$$

2) The pressure must be continuous both at the trailing edge (Kutta condition) and in the wake,

$$\Delta C_p(z = 0\pm) = 0 \quad \text{for} \quad x \geqq b_o \tag{5}$$

3) The amplitude of the disturbance approaches zero at infinity, except in the wake.

The section lift on the wing is represented more conveniently by an aerodynamic transfer function T_A, defined as $T_A \equiv C_L/(2\pi\alpha_o/\beta)$ where α_o is the angle of attack at the midchord, $\alpha_o = \hat{\alpha}\exp[i(k_y y - \omega t)]$, $\hat{\alpha} = \hat{w}/U$ and $\beta = (1-M^2)^{1/2}$.

Galilean-Lorentz Transformation

We first simplify the governing equation from the convective wave equation form to the ordinary wave equation by a technique that has been used by Amiet and Sears.[6] This technique consists of two parts: first, a Galilean transformation to a coordinate system that convects with the freestream to convert the governing equation to the ordinary wave equation; than a Lorentz transformation to reintroduce a coordinate system fixed on the wing but retaining the ordinary wave equation as the governing equation.

Let the Galilean transformation be

$$x' = x - Ut, \quad y' = y, \quad z' = z, \quad t' = t \tag{6}$$

and the Lorentz transformation be

$$X = (1/\beta)(x' + Ut'), \quad Y = y', \quad Z = z', \quad T = (1/\beta)(t' + Mx'/a) \tag{7}$$

The combined transformation is

$$X = x/\beta, \quad Y = y, \quad Z = z, \quad T = \beta(t + Mx/a\beta^2) \tag{8}$$

It can be seen that in this transformed space (X, Y, Z, T), the freestream velocity is still in the X direction with magnitude U.

The governing equation after transformation is then

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2} - \frac{1}{a^2}\frac{\partial^2\phi}{\partial T^2} = 0 \tag{9}$$

and the perturbation pressure coefficient in the transformed space is given as

$$C_p{}^T = -(2/U)[\partial\phi/\partial X + (1/U)(\partial\phi/\partial T)] \tag{10}$$

It can be shown that the perturbation pressure coefficients of the physical and the transformed spaces are related as

$$C_p = (1/\beta)C_p{}^T \tag{11}$$

The geometry of the wing-gust interaction in the transformed space can be seen more clearly through the transformation of the boundary conditions. In the transformed space, the flow tangency condition on the wing surface is

$$\phi z|_{Z=0\pm} = -\hat{w}e^{i(K_X X + K_Y Y - \Omega T)} \quad \text{for} \quad |X| \leqq b \tag{12}$$

which is transformed from Eq. (4) with $K_X = k_x/\beta$, $K_Y = k_y$ being the wave number components in the X and Y directions, and $\Omega = \omega/\beta$ being the radian frequency in the transformed space; $b = b_o/\beta$ is the semichord of the wing in the transformed space. The wave number of the gust and the wing-gust interaction angle in the transformed space is then given by

$$K = (K_X{}^2 + K_Y{}^2)^{1/2} \tag{13}$$

and

$$\Lambda^T = \tan^{-1}(K_Y/K_X) = \tan^{-1}(\beta\tan\Lambda) \tag{14}$$

From the pressure relation given by Eq. (11), the remaining two boundary conditions 1) at the trailing edge and in the wake, and 2) at infinity are the same in the transformed space as in the physical space. Therefore, we can see that in the transformed space, there exists a wing of larger chord b, interacting with a sinusoidal gust of shorter wavelength $2\pi/K$ at a smaller angle Λ^T. The amplitude of the gust remains the same as in the original space and the gust is convected with the freestream velocity U. Therefore, the character of the problem has basically not been changed through the transformation. The only difference is that the governing equation has been transformed from the convective wave equation for the original space to the ordinary wave equation for the transformed space. This transformation can be viewed as a generalization of the Prandtl-Glauert transformation for the unsteady flow. The length scale of the wing in the freestream direction is stretched by a factor $1/\beta$ as in the steady case while that of the gust is reduced by a factor of β due to the shift and change of the time scale.

In evaluating the load distribution on the wing in the physical space from the solution in the transformed space, the effect of the shift of the time scale should be treated with special care. From Eq. (8) we can see this shift is proportional to x, so that in the relation between the airload distributions in the physical and transformed space, there will be a phase shift factor that varies with chordwise locations.

From the geometry of the wing-gust interaction in both the physical space and the transformed space, we can write the differences of the pressure coefficients across the wing in both spaces in the form

$$\Delta C_p(x, y, t) = \Delta\hat{C}_p(\bar{x})e^{i(k_y y - \omega t)} \tag{15}$$

and

$$\Delta C_p{}^T(X, Y, T) = \Delta\hat{C}_p{}^T(\bar{X})e^{i(K_Y Y - \Omega T)} = \Delta\hat{C}_p{}^T(\bar{x})e^{i(k_y y - \omega t)}e^{-i\sigma\bar{x}} \tag{16}$$

where

$$\sigma \equiv sM^2/\beta^2, \quad s \equiv \omega b_o/U \quad \text{and} \quad \bar{x} = \bar{X} = x/b_o = X/b$$

Thus, from Eq. (11) we have obtained

$$\Delta\hat{C}_p(\bar{x}) = (1/\beta)\Delta\hat{C}_p{}^T(\bar{x})e^{-i\sigma\bar{x}} \tag{17}$$

and

$$T_A = (1/4\pi\hat{\alpha})\int_{-1}^{1}\Delta\hat{C}_p{}^T(\bar{x})e^{-i\sigma\bar{x}}d\bar{x} \tag{18}$$

Normalization of Variables of Inner and Outer Regions

Consider the flowfield of the transformed space being separated into two regions: the inner region in the neighborhood of the wing, and the outer region, the rest of the flowfield. The characteristic length for X and Z in the inner region should be of the order of the airfoil length; here it is chosen to be the semichord b. For the outer region, the characteristic length for X and Z should be of the order of the acoustic wavelength $\lambda_a{}^T$ of the transformed space; so we choose it to be $\lambda_a{}^T/2\pi$ or b/SM, where $S \equiv \Omega b/U$ is the reduced frequency for the transformed space.

We assume that the chord of the wing is much smaller than the acoustic wavelength. The ratio of the characteristic lengths

$$\varepsilon = SM \ll 1 \tag{19}$$

is chosen to be the small parameter for the expansions.

Since the span of the wing is infinite, the characteristic length for Y is of the order of the gust wavelength in the Y-direction for both the inner and the outer regions; we choose it to be $K_Y{}^{-1}$. The characteristic speed and time are chosen to be U and b/U, respectively. Therefore, the nondimensional variables for the inner region are

$$\bar{X} = X/b, \quad \bar{Z} = Z/b, \quad \bar{Y} = K_Y Y, \quad \bar{T} = UT/b, \quad \Phi^i = \phi^i/bU \tag{20}$$

The governing equation becomes

$$\frac{\partial^2\Phi^i}{\partial\bar{X}^2} + \frac{\partial^2\Phi^i}{\partial\bar{Z}^2} + (K_Y b)^2\frac{\partial^2\Phi^i}{\partial\bar{Y}^2} - M^2\frac{\partial^2\Phi^i}{\partial\bar{T}^2} = 0 \tag{21}$$

From the gust input boundary condition of Eq. (12), the velocity potential function for both the inner and the outer regions is of the form

$$\Phi(\bar{X}, \bar{Y}, \bar{Z}, \bar{T}) = \hat{\Phi}(\bar{X}, \bar{Z})e^{i(\bar{Y} - S\bar{T})}$$

The equation for the inner region is then reduced to

$$\frac{\partial^2\Phi^i}{\partial\bar{X}^2} + \frac{\partial^2\Phi^i}{\partial\bar{Z}^2} - (K_Y b)^2\Phi^i + (SM)^2\Phi^i = 0 \tag{22a}$$

The first, second, and fourth term in the equation are obviously of the order of 1, 1 and ε^2, respectively; whereas the order of the third term depends on the order of magnitude of $(K_Y b)$. When Λ is not too small, so that $K_Y^{-1} = k_y^{-1} \sim 0(b_o)$, then $K_Y b \sim 0(1)$ and the equation can be written as

$$\frac{\partial^2\Phi^i}{\partial\bar{X}^2} + \frac{\partial^2\Phi^i}{\partial\bar{Z}^2} + (K_Y b)^2 \frac{\partial^2\Phi^i}{\partial\bar{Y}^2} = -\varepsilon^2\Phi^i \tag{22b}$$

On the other hand, if K_Y^{-1} (or k_y^{-1}) is of the order of the acoustic wavelength, then $K_Y b \sim 0(\varepsilon)$; and the equation should be written as

$$\frac{\partial^2\Phi^i}{\partial\bar{X}^2} + \frac{\partial^2\Phi^i}{\partial\bar{Z}^2} = -\varepsilon^2\left[1-\left(\frac{\tan\Lambda^T}{M}\right)^2\right]\Phi^i \tag{22c}$$

The factor $\tan\Lambda^T/M$ is of special interest; with the transformation relations, it is easy to show that $\tan\Lambda^T/M$ being greater than, equal to or less than 1 is just equivalent to $\sin\Lambda/M$ being greater than, equal to or less than 1 in the physical space. From the geometry of the wing-gust interaction shown in Fig. 1, it can easily be seen that the Mach number $M_c = M/\sin\Lambda$ is that at which the airload convects through the air along the gust wave front. Johnson[5] showed that the problem of wing-gust interaction is an elliptic problem for $M_c < 1$, or hyperbolic problem for $M_c > 1$. Along $M_c = 1$, the problem has a transitional or transonic character (Fig. 2).

Therefore, we can see that the case $K_Y b \sim 0(1)$ applies to the portion of the elliptic region far from the transition line, which we will call the "3D region." The case $K_Y b \sim 0(\varepsilon)$ applies to the rest of the elliptic region near the transition line, on the transition line and to the whole hyperbolic region; we will call this the "quasi-2D region." The factor g^2, defined as

$$g^2 = 1 - \left(\frac{\tan\Lambda^T}{M}\right)^2 \tag{23}$$

is in turn negative, zero, and positive in the elliptic region, on the transition line and in the hyperbolic region, respectively. The terminologies of "3D region" and "quasi-2D region" are chosen because, as will be seen, the lowest order governing equation for the inner flow of these cases is either a three-dimensional or a two-dimensional Laplace equation. The 3D region and the quasi-2D region are also shown in Fig. 2.

For the outer region, the nondimensional variables are

$$X_o = \varepsilon\bar{X},\quad Z_o = \varepsilon\bar{Z},\quad Y_o = \bar{Y},\quad T_o = \bar{T},\quad \Phi^o = \phi^o/bU \tag{24}$$

The governing equation becomes

$$\varepsilon^2[\partial^2\Phi^o/\partial X_o^2 + \partial^2\Phi^o/\partial Z_o^2 + \Phi^o] - (K_Y b)^2\Phi^o = 0 \tag{25a}$$

The first three terms are obviously of order ε^2, the order of the last term, which comes from the double derivative with respect to Y, depends on the order of $(K_Y b)$. For the 3D region, $K_Y b \sim 0(1)$, so this term is of order 1, we have

$$(K_Y b)^2\Phi^o = \varepsilon^2[\partial^2\Phi^o/\partial X_o^2 + \partial^2\Phi^o/\partial Z_o^2 + \Phi^o] \tag{25b}$$

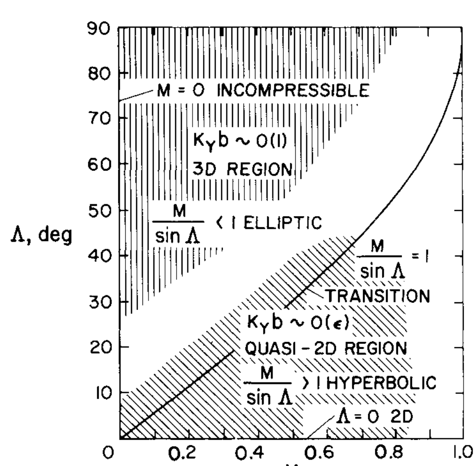

Fig. 2 3D and quasi-2D regions.

For the quasi-2D region, $K_Y b \sim 0(\varepsilon)$, so that every term of Eq. (25a) is of order ε^2, and the equation can be written as

$$\partial^2\Phi^o/\partial X_o^2 + \partial^2\Phi^o/\partial Z_o^2 + \left[1-\left(\frac{\tan\Lambda^T}{M}\right)^2\right]\Phi^o = 0 \tag{25c}$$

Matched Asymptotic Expansions for the 3D Region

From the forms of the inner and outer region governing equations given by Eqs. (22b) and (25b), we may assume the inner and outer asymptotic expansions to be of the form

$$\Phi^i = \Phi_0{}^i + \varepsilon^2\Phi_2{}^i + \ldots \tag{26a}$$

$$\Phi^o = \Phi_0{}^o + \varepsilon^2\Phi_2{}^o + \ldots \tag{26b}$$

First Outer Approximation $\Phi_0{}^o$

The first approximation for the outer region is the undisturbed flowfield if the wing were absent. The velocity potential is then a constant which we can choose to be zero.

$$\Phi_0{}^o = 0 \tag{27}$$

First Inner Approximation $\Phi_0{}^i$

The governing equation for $\Phi_0{}^i$ is

$$\frac{\partial^2\Phi_0{}^i}{\partial\bar{X}^2} + \frac{\partial^2\Phi_0{}^i}{\partial\bar{Z}^2} + (K_Y b)^2\frac{\partial^2\Phi_0{}^i}{\partial\bar{Y}^2} = 0 \tag{28}$$

If it is written in the dimensional form, it is the three-dimensional Laplace equation.

The boundary conditions on the wing surface, at the trailing edge, and in the wake are

$$\Phi^i_{0\bar{Z}}|_{\bar{Z}=0\pm} = -\hat{\alpha}\, e^{i(S\bar{X}+\bar{Y}-S\bar{T})} \quad \text{for} \quad |\bar{X}| \leqq 1 \tag{29}$$

$$\Delta C_p{}^T(\bar{Z} = 0\pm) = 0 \quad \text{for} \quad \bar{X} \geqq 1 \tag{30}$$

The boundary condition at infinity for the inner solution is obtained by applying the asymptotic matching principle:

$$\lim_{r_o/\varepsilon\to\infty} \Phi^i(r_o/\varepsilon) = \lim_{r_o\to 0}\Phi^o(r_o) = 0 \tag{31}$$

with

$$r_o = (X_o{}^2 + Z_o{}^2)^{1/2}$$

Therefore, to the lowest order, the governing equation and the boundary conditions of the inner solution are those of a three-dimensional incompressible flowfield, for which we can employ Filotas' theory. The pressure distribution and the aerodynamic transfer function of the wing are then obtained as

$$\Delta C_p(k_o, \Lambda, M; z = 0\pm) = \frac{1}{\beta}\Delta C_{p_o}(K_o, \Lambda^T; Z = 0\pm)e^{-i\sigma\bar{x}} =$$
$$4[\hat{\alpha}\, e^{i(k_y y - \omega t)}]\frac{1}{\beta}\left\{\frac{T_o(K_o, \Lambda^T)}{I_0(K_2)+I_1(K_2)}\right\}\left(\frac{1-\bar{x}}{1+\bar{x}}\right)^{1/2} e^{-(K_2+i\sigma)\bar{x}} \tag{32}$$

and

$$T_A(k_o, \Lambda, M) = T_o(K_o, \Lambda^T)\left(\frac{I_0(K')+I_1(K')}{I_0(K_2)+I_1(K_2)}\right) \tag{33}$$

where $k_o = kb_o$, $K_o = (k_o/\beta)[(\cos\Lambda/\beta)^2 + \sin^2\Lambda]^{1/2}$, $K_2 = K_o\sin\Lambda^T$, $K' = K_2 + i\sigma$, and I_0, I_1 are the first kind modified Bessel functions of the zeroth and first order. ΔC_{p_o} and T_o are the pressure difference distribution and the aerodynamic transfer function for the incompressible flow from Filotas' solution. Their expressions are given in the Appendix.

The effects of compressibility on the airload distribution are given in Eq. (32). In this expression we can see first that there is the factor $1/\beta$ as in the steady case; the next factor (included in the curved parentheses) gives an attenuation of magnitude as well as a phase shift due to both compressibility and unsteadiness. The square root factor represents the basic load distribution for a two-dimensional wing-gust interaction in incompressible flow. For three-dimensional interactions, the airload oscillation along the span and the resulting trailing vorticity will make the load distribution more concentrated toward the leading edge as shown by the factor $\exp(-K_2\bar{x})$. The last factor $\exp(-i\sigma\bar{x})$ shows a phase shift of the load distribution along the chord.

Matched Asymptotic Expansions for the Quasi-2D Region

From Eq. (22c) we can see that the lowest order governing equation for the inner region will be the two-dimensional Laplace equation with the spanwise dependence entering only as a multiplying factor in one boundary condition. In this case, both Λ and Λ^T are small while the wavelength of the gust along the span is large so that the spanwise rate of variation of the incident angle is small. Therefore, to obtain the effects of three dimensionality, we must carry the analysis to the next order. For this purpose, it is more convenient to work directly with the pressure field rather than the velocity potential. Let $P \equiv \hat{C}_p{}^T$. From Eqs. (22c) and (25c), the inner and outer flows governing equations for P can be obtained as

$$\frac{\partial^2 P^i}{\partial \bar{X}^2} + \frac{\partial^2 P^i}{\partial \bar{Z}^2} = -\varepsilon^2 g^2 P^i \tag{34}$$

and

$$\frac{\partial^2 P^o}{\partial X_o{}^2} + \frac{\partial^2 P^o}{\partial Z_o{}^2} + g^2 P^o = 0 \tag{35}$$

The boundary condition at infinity is applied to the outer solution, while the Kutta condition at the trailing edge is satisfied by the inner solution.

From Eqs. (34) and (35), we assume the inner and outer solutions are in the form

$$P^i = P_0{}^i + \varepsilon^2 \log \varepsilon P_{21}{}^i + \varepsilon^2 P_{22}{}^i + \ldots \tag{36a}$$

$$P^o = P_0{}^o + \varepsilon P_1{}^o + \varepsilon^2 P_2{}^o + \ldots \tag{36b}$$

(The need for a logarithmic term will be later confirmed.)

First Outer Approximation $P_0{}^o$

The first approximation for the outer region is the undisturbed flowfield if the wing were absent. The perturbation pressure is zero

$$P_0{}^o = 0 \tag{37}$$

First Inner Approximation $P_0{}^i$

The governing equation for $P_0{}^i$ is just the two-dimensional Laplace equation

$$\partial^2 P_0{}^i/\partial \bar{X}^2 + \partial^2 P_0{}^i/\partial \bar{Z}^2 = 0 \tag{38}$$

The inner boundary conditions include the flow tangency on the wing surface and pressure continuity at the trailing edge and in the wake. The condition at infinity for the inner solution is obtained by matching with the outer solution.

$$\lim_{r_o/\varepsilon \to \infty} P^i(r_o/\varepsilon) = \lim_{r_o \to 0} P^o(r_o) = 0 \tag{39}$$

Therefore, we can see that in the inner region to the lowest order, the problem is just the Sears' problem for the two-dimensional airfoil interacting with parallel sinusoidal gust in an incompressible flow. The pressure induced on the wing is given by

$$P_0{}^i(\bar{X}, \bar{Z} = 0\pm) = \mp 2\hat{\alpha} T_{o2D}(S)\left(\frac{1-\bar{X}}{1+\bar{X}}\right)^{1/2} \quad \text{for} \quad |\bar{X}| \leqq 1 \tag{40}$$

where T_{o2D} is the aerodynamic transfer function for this two-dimensional incompressible case

$$T_{o2D}(S) = [J_0(S) + iJ_1(S)]C^*(S) - iJ_1(S) \tag{41}$$

with $J_0(S)$, $J_1(S)$ being the zeroth and first-order Bessel functions of the first kind, and $C^*(S)$ being the conjugate of the Theodorsen function. Because we choose the time variation to be proportional to $\exp(-i\Omega T)$ rather than $\exp(i\Omega T)$ as did Sears,[1] the expression (41) is the conjugate of the function derived by him.

The pressure distribution $P_0{}^i$ for the whole inner flowfield can be obtained as an analytic function of the complex variable η

$$\eta = \bar{X} + i\bar{Z} \tag{42}$$

with a branch cut from $\eta = (-1, 0)$ to $\eta = (1, 0)$.

$$P_0{}^i(\eta) = -2\hat{\alpha} T_{o2D}(S) \operatorname{Im}\left[\left(\frac{\eta-1}{\eta+1}\right)^{1/2}\right] \tag{43}$$

Second Outer Approximation $P_1{}^o$

The governing equation for $P_1{}^o$ is a two-dimensional Helmholtz equation reduced from the wave equation,

$$\frac{\partial^2 P_1{}^o}{\partial X_o{}^2} + \frac{\partial^2 P_1{}^o}{\partial Z_o{}^2} + g^2 P_1{}^o = 0 \tag{44}$$

If $g^2 > 0$, then g is real (we choose it to be positive). The solution must satisfy the following conditions: it should be single-valued; the magnitude of the disturbance must approach zero at infinity; to this order, only outgoing waves are permitted. In addition, the outer solution to this order should match with the inner solution. The second-order outer approximation is then obtained as

$$P_1{}^o = -i\pi g \hat{\alpha} T_{o2D}(S) \sin\theta H_1{}^{(1)}(gr_o) \tag{45}$$

where $\theta = \tan^{-1}(Z_o/X_o)$, $r_o = (X_o{}^2 + Z_o{}^2)^{1/2}$

The directivity $\sin\theta$ shows the character of a stationary dipole (with the convection effect eliminated by the coordinate transformation). Hence, we may conclude that to the first order in the outer field, the effect of the wing-gust interaction is that of a line distribution of acoustic dipoles along the span with their strength proportional to the induced section lift per unit span. This would be expected on the basis of our assumption that the acoustic wavelength is much larger than the chord of the wing.

Second Inner Approximation $P_{21}{}^i$

With the first and second outer approximations given by Eqs. (37) and (45), we can now obtain the inner expansion of the outer solution to higher order as

$$P^{oi} = \hat{\alpha} T_{o2D}(S)\left[-\frac{2\sin\theta}{\bar{r}} + \varepsilon^2 \log \varepsilon g^2 \bar{r} \sin\theta + \cdots\right] \tag{46}$$

with $\bar{r} = r_o/\varepsilon$.

This confirms the necessity of a term of order $(\varepsilon^2 \log \varepsilon)$ in the inner expansion. The governing equation for $P_{21}{}^i$ is also the two-dimensional Laplace equation. It can also be written in complex variables as

$$4(\partial^2 P_{21}{}^i/\partial\eta\, \partial\eta^*) = 0 \tag{47}$$

with η^* being the conjugate of η.

The relation between the perturbation pressure and the upwash velocity of the convective gust can be obtained as

$$\hat{\alpha} = \int_{-\infty}^{\bar{X}} e^{-iS\bar{X}_1} P_{\bar{Z}}(\bar{X}_1, 0\pm)\, d\bar{X}_1 \quad \text{for} \quad |\bar{X}| \leqq 1 \tag{48}$$

with Eq. (48) already satisfied by the lowest order solution, the tangency condition for the higher order solutions should be

$$0 = \int_{-\infty}^{\bar{X}} e^{-iS\bar{X}_1} P_{\bar{Z}}(\bar{X}_1, 0\pm)\, d\bar{X}_1 \quad \text{for} \quad |\bar{X}| \leqq 1 \tag{49}$$

This condition is difficult to apply because the integral extends from the outer region into the inner region. Therefore we shall apply only the necessary condition (not sufficient)

$$P_{\bar{Z}}(\bar{X}, 0\pm) = 0 \quad \text{for} \quad |\bar{X}| \leqq 1 \tag{50}$$

to the inner solution. After the inner and the outer solutions are obtained, then we shall demonstrate that the sufficient condition is also satisfied.

With the necessary flow tangency condition (50), the Kutta condition and the wake condition being satisfied, and having the outer expansion match the corresponding part of the inner expansion of the outer solution, the second inner approximation is

$$P_{21}{}^i = \hat{\alpha} T_{o2D}(S) g^2 \operatorname{Im}[(\eta^2 - 1)^{1/2}] \tag{51}$$

Third Outer Approximation $P_2{}^o$

The governing equation of $P_2{}^o$ and its boundary condition at infinity are the same as for $P_1{}^o$. The constant coefficients can be determined by matching the expansions of the inner and outer solutions to this order. The third outer approximation is then

$$P_2{}^o = i\frac{\pi}{4} g^2 \hat{\alpha} T_{o2D}(S) \sin(2\theta) H_2{}^{(1)}(gr_o) \tag{52}$$

The directivity $\sin(2\theta)$ is that of an acoustic quadrapole, hence we may interpret P_2^o as a higher order modification of the outer field radiation.

Third Inner Approximation P_{22}^i

The governing equation for P_{22}^i is the two-dimensional Poisson's equation,

$$\partial^2 P_{22}^i/\partial \bar{X}^2 + \partial^2 P_{22}^i/\partial \bar{Z}^2 = -g^2 P_0^i \tag{53}$$

It can be written in complex variable form with P_0^i being substituted by the expression (43)

$$4(\partial^2 P_{22}^i/\partial\eta\,\partial\eta^*) = 2g^2\hat{\alpha}T_{o_{2D}}(S)\,\mathrm{Im}\left[\left(\frac{\eta-1}{\eta+1}\right)^{1/2}\right] \tag{54}$$

The particular solution of this equation can be obtained by direct integration with respect to η and η^*; to this we add solutions of the homogeneous equation. After satisfying the boundary conditions, we obtain the complete solution as

$$P_{22}^i = g^2\hat{\alpha}T_{o_{2D}}(S)\Bigg[\tfrac{1}{2}\mathrm{Im}\,[\eta^*(\eta^2-1)^{1/2}] + \bar{Z}\,Re\,\{\log[\eta+(\eta^2-1)^{1/2}]\} + \left[\left(\log\frac{g}{4}-\frac{i\pi}{2}\right) + (\gamma-\tfrac{1}{2})\right]\mathrm{Im}\,[(\eta^2-1)^{1/2}]\Bigg] \tag{55}$$

where γ is Euler's constant.

Therefore, to this order the complete inner solution is

$$P^i = \hat{\alpha}T_{o_{2D}}(S)\Bigg[-2\,\mathrm{Im}\left[\left(\frac{\eta-1}{\eta+1}\right)^{1/2}\right] + \varepsilon^2\log\varepsilon\; g^2\,\mathrm{Im}\,[(\eta^2-1)^{1/2}] + \varepsilon^2 g^2\Big\{\tfrac{1}{2}\mathrm{Im}\,[\eta^*(\eta^2-1)^{1/2}] + \bar{Z}\,\mathrm{Re}\,\{\log[\eta+(\eta^2-1)^{1/2}]\} + \left[\left(\log\frac{g}{4}-\frac{i\pi}{2}\right)+(\gamma-\tfrac{1}{2})\right]\mathrm{Im}\,[(\eta^2-1)^{1/2}]\Big\}\Bigg] \tag{56}$$

and the outer solution is

$$P^o = i\pi\hat{\alpha}T_{o_{2D}}(S)\left[-\varepsilon g\sin\theta H_1^{(1)}(gr_o) + \varepsilon^2\frac{g^2}{4}\sin(2\theta)H_2^{(1)}(gr_o)\right] \tag{57}$$

From the asymptotic behavior of the Hankel functions at infinity we can see that the pressure in the far field has the proportional relation

$$p^T \sim \frac{1}{\bar{r}^{1/2}}e^{i(g\varepsilon\bar{r}+Y-S\bar{T})} \tag{58}$$

The speed of the wave propagation is the speed of sound; the magnitude of the pressure fluctuation is proportional to $1/\bar{r}^{1/2}$. Thus in the far field, the pressure disturbance is that of an outgoing acoustic wave with conical wave fronts.

For the case $g^2 < 0$, i.e., in the elliptic region near the transition line, we choose $g' = -ig = |g|$; thus g' is real and positive. The general solution for the second and third outer approximation P_1^o, P_2^o is then of the form similar to that given earlier with the modified Bessel functions $K_m(g'r_o)$ and $I_m(g'r_o)$ replacing the Hankel functions $H_m^{(1)}(gr_o)$ and $H_m^{(2)}(gr_o)$. With the condition of vanishing disturbance at infinity, the coefficients of $I_m(g'r_o)$ must all be zero. Since the modified Bessel function and the Hankel function are related through the formula $K_m(g'r_o) = (\pi/2)i^m H_m^{(1)}(gr_o)$, the previous analysis for determining the coefficients holds for the case $g^2 < 0$. In this case, the inner solution can be obtained by replacing g in Eq. (56) by ig', while the outer solution is

$$P^o = 2\hat{\alpha}T_{o_{2D}}(S)\left[-\varepsilon g'\sin\theta K_1(g'r_o) + \varepsilon^2\frac{g'^2}{4}\sin(2\theta)K_2(g'r_o)\right] \tag{59}$$

From the asymptotic behavior of the modified Bessel functions at infinity, the pressure disturbance in the far field has the proportional relation

$$p^T \sim \frac{1}{r^{1/2}}e^{-g'\varepsilon\bar{r}}e^{i(\bar{Y}-S\bar{T})} \tag{60}$$

from which we can see that in the far field, the magnitude of the pressure fluctuation decays not only proportional to the inverse of the square root of the distance from the wing, but also exponentially; therefore, it is not an acoustic field but "pseudo sound." The disturbance propagates only in the spanwise direction with the speed of the input gust along the span, which is less than the speed of sound in this region since $g^2 < 0$.

Before we evaluate the load distribution on the wing, the tangency condition on the wing surface should be further discussed. In solving the inner solution, we have applied only the necessary condition of $P_{\bar{Z}} = 0$ (with the exception of the lowest order solution). This is not sufficient to describe the flow tangency condition. Since the original flow tangency condition of $\phi_Z^i = 0$ (for higher order solutions) will lead to $P_{\bar{Z}} = 0$ through an operation of the form.

$$\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\phi_Z^i = 0 \quad \text{for} \quad |X| \leq b, Z = 0\pm \tag{61}$$

But Eq. (61) does not uniquely result in $\phi_Z^i = 0$. ϕ_Z^i from Eq. (61) is a convective sinusoidal gust of any upwash velocity magnitude, with the correct boundary condition $\phi_Z^i = 0$ as a special case. Our argument is that if the corresponding upwash velocity magnitudes for the second and the third inner approximations P_{21}^i, P_{22}^i are not zero, then the two-dimensional incompressible flow—which the inner region is up to these orders—implies that the pressure distribution on the wing surface would have the term of $[(1-\bar{X})/(1+\bar{X})]^{1/2}$. Since this term was not present in P_{21}^i and P_{22}^i, we conclude that the flow tangency conditions

$$\phi^i_{21_Z} = \phi^i_{22_Z} = 0 \quad \text{for} \quad |X| \leq b, Z = 0\pm \tag{62}$$

are satisfied with the boundary condition of $P_{\bar{Z}} = 0$.

For the hyperbolic region ($g^2 > 0$), the difference of the pressure coefficient across the wing in the physical space is then

$$\Delta C_p(z=0\pm) = \frac{1}{\beta}\hat{\alpha}\,e^{i(k_y y-\omega t)}T_{o_{2D}}(s/\beta^2)\Bigg[4\left(\frac{1-\bar{x}}{1+\bar{x}}\right)^{1/2} - 2\varepsilon^2\log\varepsilon\; g^2(1-\bar{x}^2)^{1/2} - \varepsilon^2 g^2\Big\{\bar{x}(1-\bar{x}^2)^{1/2} + 2\left[\left(\log\frac{g}{4}-\frac{i\pi}{2}\right)+(\gamma-\tfrac{1}{2})\right](1-\bar{x}^2)^{1/2}\Big\}\Bigg]e^{-i\sigma\bar{x}} \tag{63}$$

The aerodynamic transfer function can be obtained by integrating the pressure difference over the chord

$$T_A(k_o,\Lambda,M) = T_{o_{2D}}(s/\beta^2)\Bigg[[J_0(\sigma) + iJ_1(\sigma)] - \frac{g^2}{2\sigma}\Big\{\varepsilon^2\log\varepsilon J_1(\sigma) + \varepsilon^2\left[\left(\log\frac{g}{4}-\frac{i\pi}{2}+\gamma-\frac{1}{2}\right)J_1(\sigma) - \frac{i}{2}J_2(\sigma)\right]\Big\}\Bigg] \tag{64}$$

From Eq. (63) we can see that the dominant term is the quasi-2D solution with the three-dimensional effect shown only implicitly in the reduced frequency s and in the spanwise sinusoidal variation $\exp(-ik_y y)$. The explicit three-dimensional effect on the load distribution is present only through higher order terms.

For the elliptic region near the transition line ($g^2 < 0$), the difference of pressure coefficient across the wing and the aerodynamic transfer function can be obtained by replacing g in Eqs. (63) and (64) by ig'. For the case of $g^2 = 0$ (on the transition line), the convective speed of the airload through the air along the gust wave front is just equal to the speed of sound. The problem then takes on the character of transonic flow and nonlinearity would become important. Johnson[5] has discussed this situation and concluded that the nonlinearity is not strong

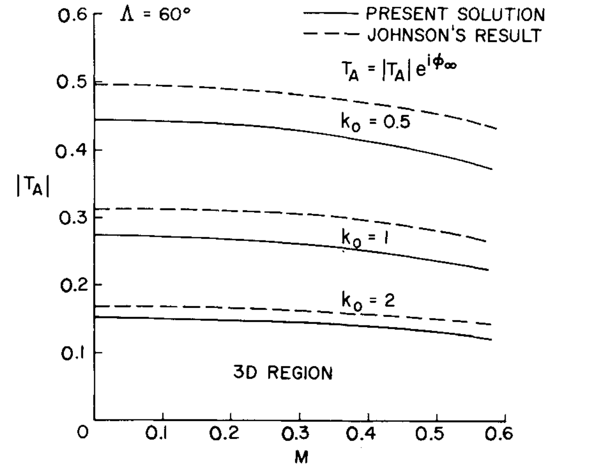

Fig. 3 Magnitude of the aerodynamic transfer function ($\Lambda = 60°$).

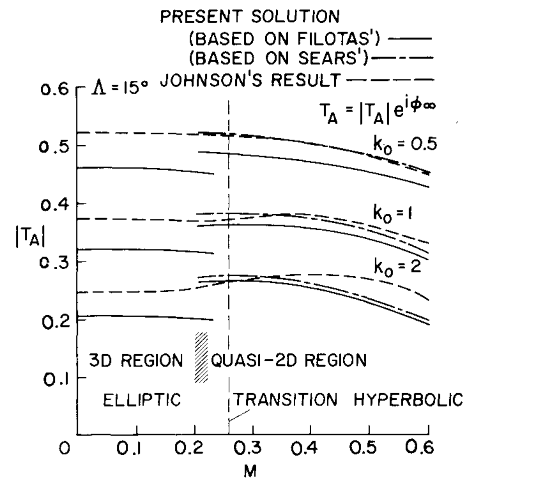

Fig. 5 Magnitude of the aerodynamic transfer function ($\Lambda = 15°$).

so that the linear solution will serve as a reasonable approximation. Based on this argument, for $g^2 = 0$, the approximation for the airload contains only the lowest order term of the expansion. This is, of course, a very rough approximation, since in this limit, the three-dimensional effects enter only as an influence factor on the gust wavelength in the chordwise direction.

Results and Discussions

The effect of compressibility for the three-dimensional wing-gust interaction problem has been shown to be very closely coupled with the geometry of the gust (the interaction angle Λ and the normalized gust wave number k_o). Compressibility influences not only the magnitude and the phase shift of the induced unsteady lift, but also the chordwise pressure distribution and the location of the center of lift.

From Filotas'[3] incompressible flow results, it can be seen that the airload becomes smaller and more concentrated toward the leading edge and the phase shift gets larger for increasing gust wave number. Re-examining the transformation we have developed, we can see that the gust in the transformed space has a shorter wavelength and a smaller interaction angle. The reduced frequency is modified by a factor $1/\beta^2$ while the spanwise component of the normalized gust wave number is modified only by $1/\beta$. Therefore the effect of compressibility will have larger influence on the phase shift than on the chordwise distribution of the airload magnitude.

In Figs. 3–6, the magnitude and the phase angle for the aerodynamic transfer function have been shown as functions of the Mach number and the gust wave number. In these results, the effects of the gust wave number and the flow Mach number on the induced airload agree with the discussions above.

In Figs. 3–6, Johnson's numerical results have also been plotted for comparison. From these figures we can see that the agreement of the present solutions and his results for the elliptic region are fairly good. In the comparison, since there exist some numerical differences between Sears',[1] Filotas',[3] and Johnson's results for the incompressible cases, only the trend of variation with the Mach number and the gust wave number should be compared. In comparison of the present solutions with Johnson's results for the hyperbolic region, the magnitude of the aerodynamic transfer functions are generally in good agreement. However, the variation of the phase angle from the present analysis and Johnson's results for the hyperbolic region have shown different trends as Mach number increases (Fig. 6). The reason for this discrepancy is yet unknown.

Since Osborne's[7] two-dimensional problem is just the limiting case of $\Lambda = 0$ for the quasi-2D case discussed here, we would expect the anomalous two-dimensional effect pointed out by Miles[8] would also show up for the three-dimensional cases with extremely small interaction angles. In such cases, the similarity rule developed here would no longer be valid.

Conclusion

A similarity rule for the three-dimensional interaction of a two-dimensional wing and a convective oblique sinusoidal gust

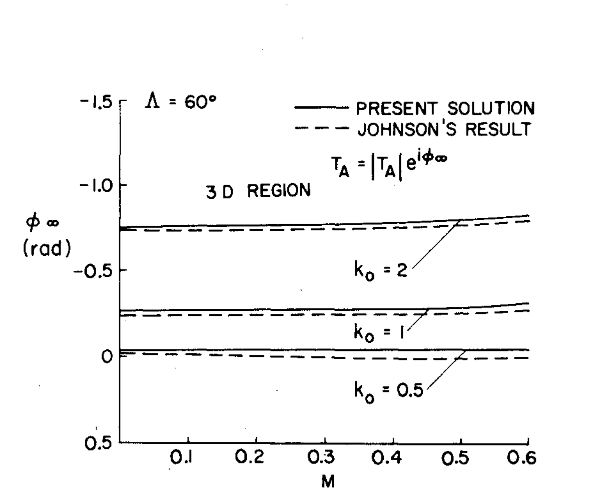

Fig. 4 Phase of the aerodynamic transfer function ($\Lambda = 60°$).

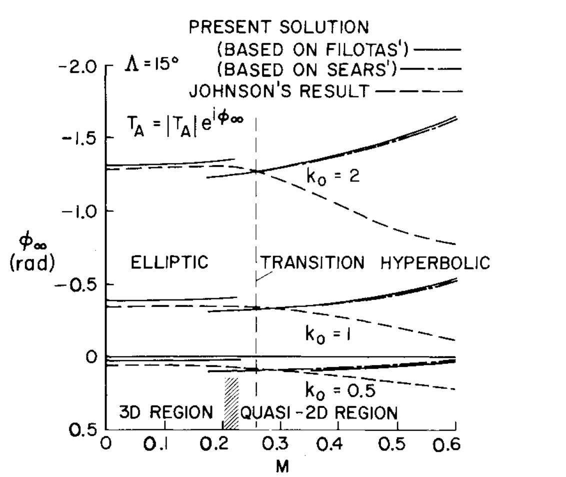

Fig. 6 Phase of the aerodynamic transfer function ($\Lambda = 15°$).

in a subsonic flow is derived. Therefore, the lift and pressure distribution induced on the wing in a compressible flow can be obtained in series form by using existing incompressible flow solutions as bases. It has been shown that the effect of compressibility depends upon the geometry of the gust, so that the similarity rule is more complicated than the Prandtl-Glauert rule for the limiting steady case. It has also been shown that the speed of the airload convection through the air is a very important factor in determining the far field radiation. Acoustic radiation results only when this speed is supersonic. Although the usefulness of the results of the analysis is somewhat limited by the requirement that the product of the flow Mach number and the reduced frequency should be small, they are expected to provide a check for more accurate numerical calculations.

Appendix: Filotas' Incompressible Flow Theory for Wing-Gust Interaction

Aerodynamic transfer function:

$$T_o(k_o, \Lambda) = \frac{\exp\left\{-ik_o\left[\cos\Lambda - \dfrac{\pi(\pi/2-\Lambda)(1+\frac{1}{2}\sin\Lambda)}{1+2\pi k_o(1+\frac{1}{2}\sin\Lambda)}\right]\right\}}{[1+\pi k_o(1+\cos^2\Lambda+\pi k_o\sin\Lambda)]^{1/2}}$$

Pressure difference distribution across the wing:

$$\Delta C_{p_o}(k_o, \Lambda; z=0\pm) = 4[\hat{\alpha}\, e^{i(k_y y-\omega t)}]\frac{T_o(k_o,\Lambda)}{I_0(k_2)+I_1(k_2)}\left(\frac{1-\bar{x}}{1+\bar{x}}\right)^{1/2} e^{-k_2\bar{x}}$$

with $k_o = kb_o$ and $k_2 = k_o \sin\Lambda$.

References

[1] Sears, W. R., "Some Aspects of Non-Stationary Airfoil Theory and Its Practical Applications," *Journal of the Aeronautical Sciences*, Vol. 8, No. 3, Jan. 1941, pp. 104–108.

[2] Graham, J. M. R., "Lifting Surface Theory for the Problem of an Arbitrary Yawed Sinusoidal Gust Incident on a Thin Aerofoil in Incompressible Flow," *Aeronautical Quarterly*, Vol. 21, Pt. 2, May 1970, pp. 182–198.

[3] Filotas, L. T., "Theory of Airfoil Response in a Gusty Atmosphere, Part I—Aerodynamic Transfer Function," UTIAS Rept. 139, AFOSR 69-2150 TR, Oct. 1969, Institute for Aerospace Studies, University of Toronto, Toronto, Ontario, Canada; also NASA SP-207, pp. 231–246.

[4] Graham, J. M. R., "Similarity Rules for Thin Airfoils in Non-Stationary Subsonic Flows," *Journal of Fluid Mechanics*, Vol. 43, Pt. 4, 1970, pp. 753–766.

[5] Johnson, W., "A Lifting Surface Solution for Vortex Induced Airloads and Its Application to Rotary Wing Airloads Calculation," ASRL TR 153-2, April 1970, Aeroelastic and Structures Research Lab., MIT, Cambridge, Mass.; also *AIAA Journal*, Vol. 9, No. 4, April 1971, pp. 689–695.

[6] Amiet, R. and Sears, W. R., "The Aerodynamic Noise of Small-Perturbation Subsonic Flows," *Journal of Fluid Mechanics*, Vol. 44, Pt. 2, 1970, pp. 227–235.

[7] Osborne, C., "Compressibility Effects in the Unsteady Interaction between Blade Rows," Ph.D. thesis, 1971, Graduate School of Aerospace Engineering, Cornell Univ., Ithaca, N.Y.; also "Unsteady Thin-Airfoil Theory for Subsonic Flow," *AIAA Journal*, Vol. 11, No. 2, Feb. 1973, pp. 205–209.

[8] Miles, J. W., "On the Compressibility Correction for Subsonic Unsteady Flow," *Journal of the Aeronautical Sciences*, Vol. 17, 1950, p. 181.

[9] Amiet, R. K., "Compressibility Effects in Unsteady Thin-Airfoil Theory," *AIAA Journal*, Vol. 12, No. 2, Feb. 1974, pp. 252–255.